Journal of Statistical Physics, Vol. 23, No. 2, 1980

Solutions of the Boltzmann Equation for Maxwell Interactions and Singular Angle-Dependent Cross Sections

H. Cornille[1] **and A. Gervois**[1]

Received November 6, 1979

We consider the spatially homogeneous and isotropic Boltzmann distribution function in the case of nonisotropic, binary cross sections inversely proportional to the relative speed of the colliding particles. Further, we allow the angle dependence of the differential cross section $\phi(\kappa)$ to be singular in the forward direction ($\kappa \to 0$). We assume $\int_0^\pi \phi(\kappa) \sin^3 \kappa \, d\kappa < \infty$, which includes the case of a Maxwellian interaction. We explicitly show how to construct the solutions of the Boltzmann equation, study their properties, and obtain for a class of solutions sufficient conditions for their existence at any positive time value. We extend the formalism to the more general case of arbitrary dimensionality. We observe an effect noticed previously by Krook, Wu, and Tjon in other models of the Boltzmann equations—namely, for special initial distributions, we find solutions which exhibit an excess of higher energy particles at later time.

KEY WORDS: Boltzmann equation; Maxwellian interaction; relaxation to equilibrium; long-tail effect.

1. INTRODUCTION

After the discovery by Bobylev,[1] Krook and Wu,[2] and Ernst[3] of particular solutions of the Boltzmann equation with a Maxwellian interaction, it appeared that in the isotropic case a general formalism could be established for the explicit construction of the solutions as well as for the determination of sufficient existence conditions.[4,5]

The key point was the possibility to determine a nonlinear differential system for the moments of the homogeneous and isotropic distribution function $f(v, t)$, where v is the velocity. This was first shown[2] for the normalized moments $M_n(t)$, which are equal to one for a pure Maxwellian

[1] CEN, Saclay, Gif sur Yvette, France.

0022-4715/80/0800-0167$03.00/0

distribution, in the particular case where $\phi(\kappa)$, the angle-dependent function of the differential cross section, is isotropic. Later Ernst,[3] in the nonisotropic case, when the long-range part of the potential is cut off such that $\int_0^\pi \phi(\kappa) \sin \kappa \, d\kappa$ is finite, extended the determination of the nonlinear system for the $M_n(t)$ as well as for the Laguerre (or Sonine) moments $b_n(t)$.

However, the realistic case corresponds to singular $\phi(\kappa)$ functions; for instance, in the Maxwellian interaction case[6] $\phi(\kappa) \simeq_{\kappa \to 0} \kappa^{-5/2}$. Here[2] we want[7] to determine the general formalism when $\phi(\kappa)$ is singular but $\int_0^\pi \sin^3 \kappa \phi(\kappa) \, d\kappa$ is finite.

In Section 2, we determine the nonlinear system satisfied by the Laguerre moments $b_n(t)$ by two methods. In the first one, we introduce the moments $M_n(t)$ as an intermediate step, whereas in the second one we consider directly the moments $b_n(t)$.

In Section 3, we establish the general properties of the Laguerre moments which can be obtained recursively from the solutions of a nonlinear system. We find two different classes, for which we introduce the arbitrary constants at infinite time or at zero time, respectively. In the first class we recover the Bobylev[1] class of particular solutions, where the $b_n(t)$ decrease like the terms of a geometrical series. However the positivity property for the sum of the Laguerre series $f(v, t)$ at $t = 0$ is not easy to control. In the second class, the number of time dependences of the $b_n(t)$ increases with n, and the positivity of $f(v, 0)$ is introduced directly with the sets $\{b_n(0)\}$. For instance, starting from the generating functional of the Laguerre generalized polynomials, we can construct a class of distribution functions $f(v, 0)$ written in closed form in such a way that the positivity appears clearly.

In Section 4 we establish sufficient conditions on the Laguerre moments at $t = 0$ such that the sums of the Laguerre series exist for $t \in [0, \infty]$. The problem is much more difficult than in the isotropic case and we consider two cases, depending upon whether or not $\phi(\kappa)$ is more singular than κ^{-2} when $\kappa \to 0$. We also give arguments concerning the positivity of $f(v, t)$ at $t > 0$ deduced from positive initial distribution function $f(v, 0)$.

In Section 5, we extend the results of the previous sections obtained in the three-dimensional space to the more general case of arbitrary dimensionality.

In Section 6, as an illustration, we present some numerical results for $f(v, t)$ corresponding to the Maxwell interactions with different initial conditions and different dimensions of the space.

In this way we observe an effect obtained previously both by Krook and Wu[2] in another Boltzmann model and by Tjon[8] in a two-dimensional case with isotropic $\phi(\kappa)$. The Tjon model is that of Tjon and Wu[9] but with different initial conditions. For different dimensional cases, we find that there

[2] A brief preliminary account of the results was presented in Ref. 7.

exist particular initial distributions such that at later time the distribution functions exhibit states with an overpopulation of higher velocity particles.

2. NONLINEAR DIFFERENTIAL SYSTEM SATISFIED BY THE BOLTZMANN MOMENTS

We start with the homogeneous Boltzmann equation and look for its isotropic solutions. We use the notations of Ref. 2, and assume for the elastic binary collisions, i.e.,

$$\mathbf{V} + \mathbf{W} = \mathbf{V}' + \mathbf{W}', v^2 + w^2 = |\mathbf{V}|^2 + |\mathbf{W}|^2 = (v')^2 + (w')^2 = |\mathbf{V}'|^2 + |\mathbf{W}'|^2$$

that the differential cross section σ is of the form $\sigma = \phi(\kappa)(|\mathbf{V} - \mathbf{W}|)^{-1}$, with κ the scattering angle, $\phi(\kappa) \geqslant 0$. We have

$$\begin{aligned}
\frac{\partial f}{\partial t}(v, t) &= \frac{1}{4\pi} \iint [f(v', t)f(w', t) - f(v, t)f(w, t)]\phi(\kappa)\, d\mathbf{W}\, d\Omega \\
d\Omega &= \sin\kappa\, d\kappa\, d\epsilon \\
(v')^2 &= v^2 + (w^2 - v^2)\sin^2 \tfrac{1}{2}\kappa + vw \sin\kappa \sin\theta \cos\epsilon \\
(w')^2 &= w^2 + (v^2 - w^2)\sin^2 \tfrac{1}{2}\kappa - vw \sin\kappa \sin\theta \cos\epsilon
\end{aligned} \tag{1}$$

where $f(v, t)$ is a function of $v = |\mathbf{V}|$ only and θ is the angle between $\mathbf{V}$ and $\mathbf{W}$.

We introduce the normalized power moments

$$M_n(t) = 2^n n!\, [(2n + 1)!]^{-1} \int f(v, t) v^{2n}\, d\mathbf{V}$$

$[M_n(t) \equiv 1$ for a Maxwellian distribution $f_{\text{Max}}(v, t) = (2\pi)^{-3/2} \exp(-v^2/2)]$ and multiply both sides of Eq. (1) by $v^{2n}\, d\mathbf{V}$ and integrate; we obtain a term proportional to $(d/dt)M_n(t)$ on the lhs and want to find on the rhs, as in the isotropic[2–10] case $\phi(\kappa) \equiv 1$, a functional of the moments $M_n(t)$. In the case where $\phi(\kappa)$ is not too singular, such that the quantity $\phi_0 = \frac{1}{2}\int \sin\kappa\phi(\kappa)\, d\kappa$ is finite, Ernst,[3] using a generalization of the Bobylev[1] method, obtained a nonlinear system for the moments $M_n(t)$. However, here we allow ϕ_0 to be infinite in such a way that only the difference of the two terms on the rhs of Eq. (1) has a meaning. Using standard techniques like those of Krook and Wu,[2] carefully taking advantage of the minus sign on the rhs, we get

$$\begin{aligned}
\frac{(2n + 1)!}{2^n n!} \frac{d}{dt} M_n = \int_0^\infty f(v, t)v^2 \left\{ \int_0^\infty f(w, t)w^2 \left\{ \int_0^\pi \frac{\phi(\kappa)\sin\kappa}{2} \left\{ \int_0^\pi \sin\theta \right.\right.\right. \\
\left.\left.\left. \times \left\{ \int_0^{2\pi} [(v')^{2n} - v^{2n}]\, d\epsilon \right\} d\theta \right\} d\kappa \right\} dw \right\} dv
\end{aligned}$$

After integration over ϵ,

$$\int [(v')^{2n} - v^{2n}]\, d\epsilon \underset{\kappa \to 0}{\simeq} \kappa^2$$

and we require the less restrictive condition

$$\int_0^\pi \sin^3 \kappa \phi(\kappa)\, d\kappa < \infty \tag{2}$$

Notice that (2) is fulfilled for Maxwellian molecules, since $\phi(\kappa) \simeq (\sin \kappa)^{-5/2}$ at small diffusion angles, although ϕ_0 does not exist.

The successive integrations are then similar to the isotropic case and we find

$$\frac{d}{dt} M_n(t) = \sum_{k=0}^{n} M_k M_{n-k} B_{k,n} C_n^{\,k}, \qquad C_n^{\,k} = n!\,[k!\,(n-k)!]^{-1}$$

$$B_{k,n} = \frac{1}{2} \int_0^\pi \phi(\kappa)(\sin \kappa)\left(\cos \frac{\kappa}{2}\right)^{2n-2k} \left(\sin \frac{\kappa}{2}\right)^{2k} d\kappa, \qquad k = 1, 2, \ldots, n \tag{3}$$

$$B_{0,n} = \frac{1}{2} \int_0^\pi \phi(\kappa)(\sin \kappa)\left[\left(\cos \frac{\kappa}{2}\right)^{2n} - 1\right] d\kappa$$

We introduce the Laguerre (or Sonine) moments $b_n(t)$:

$$b_n = \sum_{k=0}^{n} (-1)^{n+k} C_n^{\,k} M_k, \qquad M_k = \sum_{k=0}^{n} C_n^{\,k} b_k \tag{4}$$

so that f has the expansion

$$\left(\exp \frac{v^2}{2}\right)(2\pi)^{3/2} f(v, t) = \sum_{0}^{\infty} (-1)^n b_n(t) L_n^{(1/2)}(v^2/2) \tag{5a}$$

and want to obtain from Eq. (3) the corresponding system for the b_n when ϕ_0 may be infinite. The finite ϕ_0 case was considered by Ernst.[3] We differentiate (4) and obtain, using (3) and (4),

$$\frac{d}{dt} b_n = \sum_{q=0}^{n} \sum_{q'=0}^{n} b_q b_{q'} \lambda(q, q', n)$$

$$\lambda(q, q', n) = \sum_{p=q+q'}^{n} (-1)^{n+p} C_n^{\,p} \sum_{k=q}^{k=p-q'} B_{k,p} C_p^{\,k} C_k^{\,q} C_{p-k}^{\,q'}$$

In Appendix A1 it is shown that $\lambda \equiv 0$ unless $q + q' = n$, so that we finally find

$$\frac{d}{dt} b_n = \sum_{q=0}^{n} b_q b_{n-q} B_{q,n} C_n^{\,q} \tag{5b}$$

An alternative derivation of (5b) does not require the introduction of moments $M_n(t)$ as intermediate tools: substituting expansion (4a) for $f(v, t)$ into the Boltzmann equation and projecting on the nth Laguerre polynomial, we get Eq. (4b) by using explicitly the specific properties of the Laguerre $L_n^{(\alpha)}(x)$. The proof is sketched in Appendix A2.

3. GENERAL PROPERTIES OF THE SOLUTIONS OF THE NONLINEAR SYSTEM

In Eqs. (5a) and (5b) we introduced the constraints due to the conservation laws of mass ($M_0 \equiv 1, b_0 \equiv 1$) and energy ($M_1 \equiv 1, b_1 \equiv 0$). Define $a_n = b_{n+2}$, $n \geqslant 0$, for the remaining nontrivial Laguerre moments, $x = v^2/2$ as a new variable, and

$$F(x = v^2/2, t) = [\exp(v^2/2)](2\pi)^{3/2} f(v, t)$$

as a new function. We obtain

$$F(x, t) = 1 + \sum_{0}^{\infty} (-1)^n a_n(t) L_{n+2}^{(1/2)}(x) \tag{6a}$$

$$\frac{d}{dt} a_n(t) + \beta_n a_n(t) = \sum_{m=0}^{n-2} a_m a_{n-m-2} B_{m+2,n+2} C_{n+2}^{m+2} \tag{6b}$$

with

$$\begin{aligned} \beta_n &= -B_{0,n+2} - B_{n+2,n+2} \\ &= \frac{1}{2} \int_0^{\pi} d\kappa\, \phi(\kappa)(\sin \kappa) \left[1 - \left(\cos \frac{\kappa}{2} \right)^{2n+4} - \left(\sin \frac{\kappa}{2} \right)^{2n+4} \right] \end{aligned}$$

The problem is reduced to the resolution of the nonlinear system (6b) and the substitution of the solutions $a_n(t)$ into the Laguerre expansion (6a) to build $F(x, t)$.

3.1. Introduction of the Moments of $\phi(\kappa)$

We remark that all the $B_{m,n}$ are not independent. It is convenient to define a set of moments of $\phi(\kappa)$ in such a way that the coefficients of (6b) can be rewritten as a linear combination of them. We define

$$\frac{2^{2n}(n!)^2}{(2n+1)!} \phi_n = \frac{1}{2} \int_0^{\pi} (\sin \kappa)^{2n+1} \phi(\kappa)\, d\kappa, \qquad n = 0, 1, 2, \ldots \tag{7}$$

normalized in such a way that $\phi_n \equiv 1$ if $\phi \equiv 1$. Now, ϕ_0 does not appear in Eq. (6b), whereas the condition (2) implies $\phi_n < \infty\ \forall n \geqslant 1$. In Eq. (6b) the coefficients β_n of a_n, $B_{m+2,n+2} + B_{n-m,n+2}$ of $a_m a_{n-2-m}$, and $B_{n/2+1,n+2}$ of $a^2_{n/2-1}$ if n is even, can be written in terms of these ϕ_n, and in Appendix B1 we obtain

$$\beta_n = -(n+2) \sum_{p=1}^{[n/2]+1} (-1)^p \frac{(n+1-p)!}{(n+2-2p)!} \frac{p!}{(2p+1)!} \phi_p$$

$$B_{m+2,n+2} + B_{n-m,n+2} = (n-2m-2) \sum_{p=0}^{[n/2]-m-1} \frac{(-1)^p}{p!\, 2^{2p}} \times \frac{(n-2m-p-3)!}{(n-2m-2-2p)!} \frac{[(p+m+2)!]^2}{(2p+2m+5)!} \phi_{p+m+2}$$

$$B_{n/2+1,n+2} \doteq \frac{[(n/2+1)!]^2}{(n+3)!} \phi_{n/2+1}, \qquad n \text{ even} \tag{8}$$

with $[n/2] = n/2$ if n is even and $[n/2] = (n-1)/2$ if n is odd. We remark that ϕ_1 appears only in β_n, i.e., in the linear part of Eq. (6b).

Let us call $L_n(a_n)$ the linear part of the lhs of (6b) and $N_n(a_p), p \leqslant n-2$, the nonlinear part of the rhs. From the explicit expression of the coefficients written down in Eq. (8) we see that $L_n(a_n)$ depends on $\phi_1, \phi_2, \ldots, \phi_{[n/2]+1}$, whereas N_n depends on $\phi_2, \phi_3, \ldots, \phi_{[n/2]+1}$.

Due to the recursive character of the system (6b), where only the a_p with $p \leqslant n$ appear on the rhs, we see that $a_n(t)$, the nth coefficient of the Laguerre expansion (6a), depends only on the first $[n/2]+1$ moments $\phi_1, \ldots, \phi_{[n/2]+1}$ of $\phi(\kappa)$. Consequently, if in Eq. (6a) we stop the expansion at a fixed nth term, then the approximate $F(x,t)$ solution is the same for all $\phi(\kappa)$ having the same moments $\phi_1, \ldots, \phi_{[n/2]+1}$.

Let us see how the simplest particular Bobylev solution,[1] which is the same as that of Krook and Wu,[2] can be deduced from considerations on these moments. That solution depends only on ϕ_1; consequently, putting to zero all the coefficients of $\phi_p, p \neq 1$, we get $N_n(a_p) \equiv 0$. For this solution, the nonlinear system degenerates to a linear one and from $L_n(a_n) = 0$ we obtain

$$a_n(t) = a_n(0) \exp[-\tfrac{1}{6}(n+2)t\phi_1]$$

where the $a_n(0)$ are unknown. Returning to Eq. (6b), the coefficients of $\phi_2, \phi_3, \ldots$ must be identically zero. From Eq. (8) and requiring that the coefficient of ϕ_2 in $L_n(a_n)$ and $N_n(a_p)$ are the same, we get

$$a_n(0) = -a_0(0)a_{n-2}(0)(n+1)(n-1)^{-1}$$

Similarly, since the coefficient of ϕ_3 is the same in $L_4(a_4)$ and $N_4(a_p)$, we obtain $4a_0(0)^3 + a_1(0)^2 = 0$. Defining $a_0(0) = -c^2$, $c > 0$, and $a_1(0) = \pm 2c^3$, we deduce

$$a_n(0) = (\pm)^n(-1)^{n+1}c^{n+2}(n+1)$$

with still the ambiguity with regard to the plus or minus sign. Summing the Laguerre series (6a), we obtain for the sum

$$(1 \mp c)^{-5/2}\left[\left(1 \mp \frac{5c}{2}\right) \pm \frac{v^2}{2} c(1 \mp c)^{-1}\right] \exp\left[\frac{v^2}{2} c(c \mp 1)^{-1}\right]$$

so that at $t = 0$ the positivity requires the plus sign and $0 < c \leqslant 2/5$. Finally we have the simple Bobylev–Krook–Wu particular solution, which is the unique solution depending only on ϕ_1. Later we shall deduce the other Bobylev solutions.

3.2. General Structure of the Solutions of Eq. (6b)

For $n = 0$ and 1, we integrate directly and get $a_0(t) = a_0(0) \exp(-\frac{1}{3}\phi_1 t)$ and $a_1(t) = a_1(0) \exp(-\frac{1}{2}\phi_1 t)$; for $n \geqslant 2$ we take into account the nonlinear part and obtain

$$\begin{aligned} a_2(t) &= a_2(0) \exp[-(\tfrac{2}{3}\phi_1 - \tfrac{1}{15}\phi_2)t] \\ &\quad + 3a_0{}^2(0)\{\exp[-(\tfrac{2}{3}\phi_1 - \tfrac{1}{15}\phi_2)t] - \exp(-\tfrac{2}{3}\phi_1 t)\} \end{aligned}$$

and so on. Appearing explicitly in the solutions are not only the moments ϕ_p, but also arbitrary integration constants. The general solution can be written differently, depending upon whether we integrate from ∞ or 0. Let us define $\tilde{a}_n(t)$:

$$\tilde{a}_n(t) = (\exp \beta_n t) \sum_0^{n-2} a_m a_{n-2-m} B_{m+2,n+2} C_{n+2}^{m+2}$$

and integrate Eq. (6b). We get for $a_n(t)$ two possible expressions:

$$a_n(t) = (\exp -\beta_n t)\left[\bar{a}_n + \int_\infty^t \tilde{a}_n(t')\, dt'\right] \tag{9a}$$

$$a_n(t) = (\exp -\beta_n t)\left[a_n(0) + \int_0^t \tilde{a}_n(t')\, dt'\right] \tag{9b}$$

where the integration constants $\bar{a}_n$ and $a_n(0)$ satisfy the relation $\bar{a}_n - a_n(0) = \int_0^\infty \tilde{a}_n(t)\, dt$. The validity of Eq. (9a) requires $\lim_{t\to\infty} \tilde{a}_n(t) \to 0$ and we shall prove this property. If it is true, then $\bar{a}_n = \lim_{t\to\infty} a_n(t) \exp \beta_n t$. The general solution of $a_n(t)$ depends on n arbitrary constants, which we can choose among $\{a_p(0)\}, p = 0,\ldots, n-2$, and $\{\bar{a}_p\}, p = 0,\ldots, n-2$, with the rule that for p fixed we take either $a_p(0)$ or $(\bar{a}_p)$. If we retain only one constant different from zero for some $n = n_0$, either $a_{n_0}(0)$ or $\bar{a}_{n_0}$, then we define a particular solution of the system. In this way we define two bases. In the first one, each solution element is defined by the associated constant $a_n(0)$ and we call it the fundamental positive solution because we easily control the positivity of $F(x, 0)$. In the second basis, each solution element is defined by $\bar{a}_n$ and we call it the Bobylev fundamental solution because it was found previously by Bobylev[1] starting from an entirely different point of view of invariance group theory.

3.3. $a_n(t)$ for n Fixed Decreases at Least Like $\exp(-\beta_n t)$

From the definition (6b) of β_n we see that $\beta_0 > 0$ and $\{\beta_n\}$ is an increasing positive sequence. Now, for $n = 0, 1$, a_n decreases at least like $\exp(-\beta_n t)$ and assuming that this property holds for $p = 0, 1, \ldots, n-2$ we want to prove it for n. In Appendix B2 it is shown that $\beta_n - \beta_m - \beta_{n-m-2} < 0$ $\forall m \in [0, n-2]$. It follows that in (9a) and (9b), $\tilde{a}_n$ decreases at least like $\exp[-t(\beta_m + \beta_{n-m-2} - \beta_n)]$ and in Eq. (9a) we can integrate $\tilde{a}_n$ when $t \to \infty$. It follows that the property is true for n. Consequently $\lim_{t\to\infty} a_n(t) = 0$ and for x fixed, $\lim_{t\to\infty} F(x, t) = 1$, ensuring the Maxwellian behavior for $f(v, t)$.

3.4. The Solutions Defined by $\{a_n(0)\}$

Due to $\beta_n > 0$ and $B_{m+2,n+2} > 0$, the study is very similar to the isotropic case.[4,5] The *fundamental positive solutions* are such that only one $a_{p-1}(0) \neq 0$, p integer $\geqslant 1$. These solutions are not necessarily positive for any $t \geqslant 0$ value. However, the Laguerre series for $F(x, 0)$ has only one term and we find easily the condition on $a_{p-1}(0)$ such that $F(x, 0) > 0$. This requires at least $a_{p-1}(0) > 0$. We define $\gamma(k) = (k+1)(p+1)$, $k = 0, 1, \ldots,$ and $a_{n=\gamma(k)-2} = c_k$; substituting into Eq. (9), we find for the $a_n(t)$ nonidentically zero

$$\begin{aligned} c_0(t) &= a_{p-1}(0)\exp(-t\beta_{p-1}) \\ c_k(t) &= \exp(-t\beta_{\gamma(k)-2}) \\ &\quad \times \left\{ \int_0^t \exp(t'\beta_{\gamma(k)-2}) \left[\sum_0^{k-1} c_q(t') c_{k-q-1}(t') B_{\gamma(q),\gamma(k)} C_{\gamma(k)}^{\gamma(q)} \right] dt' \right\} \end{aligned} \tag{10a}$$

and the $c_k(t)$ can be obtained recursively from $c_0(t)$. When k increases, the number of terms in c_k with different time dependences increases also. The Laguerre expansion becomes

$$\begin{aligned} F(x, t) &= 1 + \sum c_k(t)(-1)^{\gamma(k)} L_{\gamma(k)}^{(1/2)}(x) \\ F(x, 0) &= 1 + a_{p-1}(0)(-1)^{p-1} L_{p+1}^{(1/2)}(x) \end{aligned} \tag{10b}$$

and so whereas at $t = 0$ we have only one $L_n^{(1/2)}(x)$, as soon as t becomes positive, we have an infinite number of $L_n^{(1/2)}(x)$.

(i) If $a_{p-1}(0) > 0$, then by induction we find $c_k(t) > 0$ and $M_n(t) > 0$. Further, for $k \neq 0$ we have dc_k/dt positive at $t = 0$ and tending to 0^- when t

$\to \infty$, whereas $c_0(t)$ is always decreasing. Finally we have $F(x, 0) > 0$ for x large enough.

(ii) If $a_{p-1}(0) < 0$ we obtain $c_k(t)(-1)^{k+1} > 0$ and for $k \neq 0$ the derivative $(-1)^{k+1}(d/dt)c_k(t)$ is positive at $t = 0$ and tends to 0^- when $t \to \infty$. For x sufficiently large $F(x, 0) < 0$ and the positivity is violated.

Can we define *an infinite mixing* of these fundamental solutions in such a way that we easily control the positivity of $F(x, t)$ at $t = 0$? An easy way, as in the isotropic case,[5] is to start with the generating functional of the generalized Laguerre polynomials

$$\sum z^n L_n^{(\alpha)}(x) = (1 - z)^{-1-\alpha} \exp[xz/(z - 1)]$$

(Although in this section we consider only $\alpha = 1/2$, the results that we present are valid for other α values and this freedom will be used later when we consider expansions of Laguerre polynomials with $\alpha \neq 1/2$.) We first remark that any z derivative has a sum written down in terms of Laguerre polynomials of argument $x/(1 - z)$:

$$\sum_{n \geqslant q} C_n{}^q z^n L_n^{(\alpha)}(x) = (1 - z)^{-(\alpha+1)}\left(\exp \frac{xz}{z - 1}\right)\left(\frac{z}{1 - z}\right)^q L_q^{(\alpha)}\left(\frac{x}{1 - z}\right)$$

By linear combinations of such derivatives of arbitrary order (such that the coefficient of $L_0^{(\alpha)}$ is one and that of $L_1^{(\alpha)}$ is zero) we get families of sums of Laguerre polynomials written in closed form:

$$\begin{aligned} F(x, 0) &= \sum_0^{\infty} L_n^{(\alpha)}(x) z^n \left(\sum_{p=0}^{q} d_p C_n{}^p\right) \\ &= (1 - z)^{-1-\alpha} \exp\left(\frac{xz}{z - 1}\right) \sum_{p=0}^{q} d_p \left(\frac{z}{1 - z}\right)^p L_p^{(\alpha)}\left(\frac{x}{1 - z}\right) \end{aligned} \tag{11}$$

where $\alpha = 1/2$, $d_0 = 1$, $d_1 = -1$, whereas z and d_p, $p > 1$, are arbitrary but subject to the constraint that the rhs of Eq. (11) must be positive. For instance, if $d_p \equiv 0$ for $p > 1$ we obtain the particular Bobylev–Krook–Wu solution[1,2] at $t = 0$. Another simple family can be obtained by subtracting the generating functional for two different z values

$$\begin{aligned} &1 - z_1 z_2 \sum_{n \geqslant 2} L_n^{(\alpha)}(x) \sum_{p=0}^{n-2} z_1{}^p z_2^{n-p-2} \\ &= (z_2 - z_1)^{-1}\left[\frac{z_2}{(1 - z_1)^{1+\alpha}} \exp \frac{xz_1}{z_1 - 1} - \frac{z_1}{(1 - z_2)^{1+\alpha}} \exp \frac{xz_2}{z_2 - 1}\right] \end{aligned} \tag{12}$$

and so on.

3.5. Solutions Defined by $\{\bar{a}_n\}$

The *fundamental solutions* (or Bobylev solutions) are such that only one $\bar{a}_{p-1} \neq 0, p$ integer $\geqslant 1$. These solutions were called "pure solutions" in the isotropic two-dimensional case.[4] They have very interesting properties: (i) they were obtained by Bobylev through group-invariance considerations; (ii) for any n we have only one time-dependent term, decreasing like the term of a geometrical series.

However, their drawback is that the initial value $F(x, 0)$ is not easily expressed in terms of their arbitrary parameters. Consequently for $F(x, t)$, the positivity property (or a violation of it) is not easily established. We recall that in the isotropic two-dimensional case,[4] for these pure solutions or for a *finite* mixing of such solutions we have *numerically* found only one solution, the Bobylev–Krook–Wu one, not violating positivity. Here, the fundamental difference, in the nonisotropic case, is that we have a new degree of freedom. For each pair of $(2q)$th and $(2q+1)$th equations of the system (6b) a new moment ϕ_q appears. One may hope that other positive solutions may be obtained, at least for particular values of the ϕ_q, though they may not be chosen completely at random, since they are moments of a positive function $\phi(\kappa)$. The following discussion shows the difficulty of such a program of research; at present we are not able to construct explicitly any positive solution (except the BKW) starting from a *finite number* of $\bar{a}_n$. We begin with these *fundamental solutions*; we consider $\gamma(k)$ as defined previously, define $a_{n=\gamma(k)-2} = c_k$, and substitute into Eq. (9):

$$c_k(t) = \delta_k \exp[-(k+1)\beta_{p-1}t], \qquad c_0 = \bar{a}_{p-1} \exp(-\beta_{p-1}t) \tag{13a}$$

$$\begin{gathered} [-(k+1)\beta_{p-1} + \beta_{\gamma(k)-2}]\delta_k = \sum_{q=0}^{k-1} \delta_q \delta_{k-q-1} C_{\gamma(k)}^{\gamma(q)} \beta_{\gamma(q),\gamma(k)} \\ F(x,t) = 1 + \sum_k (-1)^{\gamma(k)} c_k(t) L_{\gamma(k)}^{(1/2)}(x) \end{gathered} \tag{13b}$$

We consider now the *mixing of two pure solutions* where only $\bar{a}_{p_1-1} \neq 0$ and $\bar{a}_{p_2-1} \neq 0$, $p_2 > p_1 \geqslant 1$. If we put

$$\begin{gathered} a_n = \sum_r d_n^{(r)} \exp\left[-t\left(\beta_{p_1-1}\frac{n+2}{p_1+1} + \frac{\theta}{p_1+1}r\right)\right] \\ \theta = \beta_{p_1-1}(p_2+1) - \beta_{p_2-1}(p_1+1) \end{gathered} \tag{14a}$$

where the summation on r has an n-dependent number of terms, and substitute into Eq. (9), we find that the $\{d_n^{(r)}\}$ can be determined recursively

$$\left(\beta_n - \beta_{p_1-1}\frac{n+2}{p_1+1} - \frac{\theta}{p_1+1}r\right)d_n^{(r)} = \sum_{\substack{M+Q=n-2 \\ s+t=r}} B_{M+2,n+2} C_{n+2}^{M+2} d_M^{(s)} d_Q^{(t)} \tag{14b}$$

If $\theta \neq 0$, the number of different time dependences in $a_n(t)$ increases and cannot stay finite when $n \to \infty$. Now we consider the possibility $\theta = 0$ or $\beta_{p_1-1}(p_2+1) = \beta_{p_2-1}(p_1+1)$, which can be rewritten with the integral representation of the β_n:

$$\frac{1}{2}\int_0^\pi d\kappa\, \phi(\kappa)(\sin\kappa)\left\{(p_1+1)\left[1-\left(\cos\frac{\kappa}{2}\right)^{2(p_2+1)} - \left(\sin\frac{\kappa}{2}\right)^{2(p_2+1)}\right]\right.$$
$$\left. -(p_2+1)\left[1-\left(\cos\frac{\kappa}{2}\right)^{2(p_1+1)} - \left(\sin\frac{\kappa}{2}\right)^{2(p_1+1)}\right]\right\} \equiv 0$$

Studying the variation of the bracket, we find that it is zero identically for $p_1 p_2 = 2$, otherwise it always has the same sign. It follows that for $p_1 p_2 = 2$ or $p_1 = 1, p_2 = 2$, which is called the degenerate case by Bobylev,[1] there exists only one time dependence for a_n, as for the pure solution. On the contrary, for the other case $p_1 p_2 \neq 2$, due to the positivity of $\phi(\kappa)$, then the integrand $\phi(\kappa)\sin\kappa$ multiplied by the bracket always has the same sign and $\theta \equiv 0$ is impossible. We see the role of the positivity of $\phi(\kappa)$, which forbids the type of solution allowed by the algebraic structure of Eqs. (14). We consider now $p_1 = 1$ and $p_2 = 2$, where $\theta \equiv 0$ in Eqs. (14). We find

$$a_n(t) = \delta_n \exp\left[-t\phi_1 \frac{n+2}{6}\right]$$
$$a_0(t) = \delta_0 \exp\left(-t\frac{\phi_1}{3}\right), \qquad a_1(t) = \delta_1 \exp\left(-t\frac{\phi_1}{2}\right) \tag{15a}$$

$$\left(\beta_n - \phi_1 \frac{n+2}{6}\right)\delta_n = \sum_{m=0}^{n-2} \delta_m \delta_{n-m-2} B_{m+2,n+2} C_{n+2}^{m+2}, \qquad n \geqslant 2 \tag{15b}$$

In Eq. (15b) the coefficients of δ_n on the lhs as well as the coefficients of $\delta_m \delta_{n-m-2}$ on the rhs are independent of ϕ_1 and thus the same property holds for δ_n. If $4\delta_0^3 + \delta_1^2 = 0$, we know that δ_n is in fact also independent of the other ϕ_p and this case corresponds to the particular Bobylev–Krook–Wu solution. If this relation between δ_0 and δ_1 does not hold, then δ_n, $n \geqslant 4$, depends on $\phi_p, p \geqslant 2$, and we get a class of solutions. Can we manage the δ_0 and δ_1 value as well as the $\phi_p, p \geqslant 2$, in such a way that $F(x, 0) > 0$ and $\phi(\kappa) > 0$? The difficulty for $F(x, 0)$ is due to the oscillations of the Laguerre polynomials in such a way that it is not easy to characterize the properties of a set (δ_n) leading to a sum of Laguerre polynomials positive for $x > 0$.

4. SUFFICIENT CONDITIONS FOR THE EXISTENCE OF $F(x, t)$ FROM INITIAL CONDITIONS ON THE SET $\{a_n(0)\}$

Formally, Eqs. (5) or (6) generate for every time, solutions of the Boltzmann equation in terms of their Sonine expansion. However, the only solution exhibited up to now is the Bobylev–Krook–Wu (BKW) function, and one may wonder whether the Sonine expansion really has a meaning.

In Sections 4.1–4.2, we show that there is an infinite number of initial conditions such that the norm of $f(v, t)$ in the Hilbert space spanned by the Laguerre $L_n^{1/2}(v^2/2)$ remains uniformly bounded for every (finite or infinite) time t. Then, there is an infinite number of solutions which do not relax to the Maxwellian like the BKW. As a consequence, the Sonine expansion is well adapted for numerical purposes.

Another point which is rarely mentioned in the literature is that of the positivity of the solutions: if $F(x, 0)$ is a positive function, does $F(x, t)$ remain positive for all positive times? In Section 4.3, we give arguments for this; it ensures that there is an infinite number of positive solutions different from the BKW one, and in Section 6 we shall give some examples.

Taking into account the normalization of the generalized orthogonal Laguerre polynomials, the problem is to find conditions on the set $\{a_n(0)\}$ so that

$$\tilde{N}(t) = \sum_{n=n_0}^{\infty} a_n{}^2(t)\lambda_n < \infty, \qquad \forall t \in [0, \infty] \tag{16}$$

where $\lambda_n = \Gamma(n + 7/2)/\Gamma(n + 3)$ is the normalization constant of the $(n + 2)$th Laguerre $L_{n+2}^{(1/2)}$. Comparing with the isotropic case,[5] we shall operate with a slight modification. Instead of $\tilde{N}(t)$ let us define

$$N(t) = \sum_{n=n_0}^{\infty} |a_n(t)|\sqrt{\lambda_n} \tag{17}$$

where n_0 is such that $a_n(0) = 0$ if $n < n_0$, $a_{n_0}(0) \neq 0$, and then $a_n(t) \equiv 0$ if $n < n_0$. We try to find conditions on $N(0)$ so that $N(t) \leqslant \infty$ $\forall t \in [0, \infty]$. Noticing that $\tilde{N}(t) < N^2(t)$, we see that we can find conditions such that $\tilde{N}(t)$ remains bounded for any t value. We allow $\phi(\kappa)$ to be singular when $\kappa \to 0$,

$$\phi(\kappa) < c[\sin(\kappa/2)]^{-2\eta}, \qquad \eta < 2, \quad \phi_1 < \infty \tag{18}$$

Note that we can always take $c = 1$ with a new definition of the time variable $t \to tc$ in Eq. (1) and find for the coefficients of the nonlinear part of the system (6b)

$$\begin{aligned} B_{m+2,n+2}C_{n+2}^{m+2} &< c\,\frac{\Gamma(n+3)}{\Gamma(n+4-\eta)}\,\frac{\Gamma(m+3-\eta)}{\Gamma(m+3)} \\ &\leqslant \text{const}_1\, n^{\eta-1} + \text{const}_2 \end{aligned} \tag{19}$$

We have two cases, depending on whether $\eta \leqslant 1$, with bounded coefficients of the nonlinear part, or $\eta > 1$, where they increase at most like $n^{\eta-1}$. Consequently, we consider two types of bounds.

4.1. $\eta \leqslant 1$

We follow essentially the method of the isotropic case[4,5] and obtain the following result in Appendix C2: if

$$N(0) \leqslant \beta_{n_0}(n_0 + 2)(\Lambda_{2n_0+2})^{-1} \tag{20}$$

where Λ_n is a constant defined in Appendix C1, then we have for $N(t)$ an explicit upper bound for $N(t)$ such that $N(t) \leqslant N(0)$. For each n_0 value we can calculate explicitly the sufficient upper bound to be satisfied by $N(0)$. We also find a bound (C5) on $|a_n(t)|$ which decreases at least like $\exp(-\beta_{n_0}t)$.

4.2. $\eta < 2$

In this case the coefficients of the nonlinear part of (6b) increase if $\eta > 1$, whereas β_n also increases if $\eta \geqslant 1$. We assume

$$0 < c_{\text{inf}} < \phi(\kappa)[\sin(\kappa/2)]^{2\eta} < c_{\text{sup}} \tag{18$'$}$$

where either c_{inf} or c_{sup} can be chosen 1 following the same remark as above, $t \rightarrow c_{\text{inf}}t$ or $t \rightarrow c_{\text{sup}}t$. We find

$$c_{\text{inf}} \leqslant \beta_n \frac{(\eta - 1)(n + 3 - \eta)}{n + 2} \left[\frac{\Gamma(n + 2)\Gamma(2 - \eta) - 1}{\Gamma(n + 3 - \eta)}\right]^{-1} \leqslant c_{\text{sup}} \tag{21}$$

or equivalently for $\eta \neq 1$,

$$\text{const}_1 n^{\eta - 1} + \text{const}_2 \leqslant \beta_n \leqslant \text{const}_3 n^{\eta - 1} + \text{const}_4$$

and β_n increases like $\log n$ for $\eta = 1$. Consequently the coefficient $B_{m+2,n+2}C^{m+2}_{n+2}$ of the nonlinear part of the differential system (6b) does not increase with n more than β_n and this fact gives us the possibility to control the growth of the solutions.

Let us recall that for η fixed, the β_n are positive and increasing with n. We define two constants r, $0 < r < 1$, and R_{n_0} such that

$$B_{m+2,n+2}C^{m+2}_{n+2}/\beta_n < R_{n_0} \qquad \forall n \geqslant 2n_0 + 2, \quad \forall m \in [n_0, n - 2] \tag{22}$$

which can be explicitly determined from both the upper bound in (19), c being c_{sup}, and the lower bound in (21) containing c_{inf}. Of course R_{n_0} depends on η. From the explicit representation, Eq. (9b), of the solutions $a_n(t)$ we find the following result in Appendix C3: if $N(0)$ is such that

$$N(0) \leqslant (1 - r)/4\Lambda_{2n_0+2}R_{n_0} \tag{23}$$

it follows that $N(t)$ is bounded

$$N(t) \leqslant [\exp(-r\beta_{n_0}t)] \frac{1 - [1 - 4R_{n_0}N(0)\Lambda_{2n_0+2}(1 - r)^{-1}]^{1/2}}{2R_{n_0}\Lambda_{2n_0+2}(1 - r)^{-1}} \tag{24}$$

and consequently $N(t) < \text{const} \times N(0)$ for any $t \in [0, \infty]$. Further, it is shown that

$$|\sqrt{\lambda_n} a_n(t)| < [\exp(-r\beta_n t)] K_n$$

with K_n determined in Appendix C3. Notice that the above proof fails if $c_{\text{inf}} = 0$, because R_{n_0} is not finite. It implies for $\phi(\kappa)$ a very complicated oscillating regime near $\kappa = 0$, which does not occur in the practical case. Now for $\eta \leqslant 1$ we have also the previous bound, which does not require c_{inf} and which holds even if $c_{\text{inf}} = 0$. Notice also that for $\eta < 1$, the positive increasing sequence β_n tends to a limit and so the time dependence for the bounds on $|a_n(t)|$ tends to $\exp(-\text{const} \times t)$. On the contrary, for $\eta \geqslant 1$, the same time-dependent bound for $|a_n(t)|$ has a coefficient β_n which tends to infinity when n goes to infinity.

We emphasize that our conditions on $N(0)$ are only *sufficient* conditions for the Sonine expansion to converge. Presumably, finer results may be obtained by less drastic majorations. But it is enough to prove that the set of solutions of the Boltzmann equation (BE) with convergent Sonine expansion is not empty, and actually is infinite. Further, in the next section we give arguments showing that the BE carries positivity; if we start with a positive distribution function with conditions (20) or (23) being satisfied, it remains defined and positive when time is increasing. Thus, there exist infinitely many positive solutions of the BE and they do not relax to the Bobylev–Krook–Wu distribution.[1,2]

4.3. Some Results Concerning the Positivity Property of $f(v,t)$ for $t \geqslant 0$

We want to present arguments concerning the positivity of $f(v, t)$ for $t > 0$ and v finite if we start with a smooth distribution $f(v, 0) > 0$ [$f(v, 0)$ being finite for finite v values]. We assume that the possible singularity of $\phi(\kappa)$ is only at $\kappa = 0$, so that

$$\frac{1}{2} \int_{\kappa_0}^{\pi} \phi(\kappa) \sin \kappa \, d\kappa = \phi_{\kappa_0}$$

is finite for κ_0 finite but can tend to $+\infty$ if $\kappa_0 \to 0$. Equation (1) can be written, after integration from 0 to t,

$$\begin{aligned}
&[\exp(\phi_{\kappa_0} t)] f(v, t) \\
&\quad = f(v, 0) + \left\{ \int_0^t [\exp(\phi_{\kappa_0} t')][A_{\kappa_0}(v, t') + B_{\kappa_0}(v, t') \, dt' \right\} \qquad (25)
\end{aligned}$$

$$A_{\kappa_0} = \frac{1}{4\pi} \int_{\kappa_0}^{\pi} \phi(\kappa)(\sin \kappa) \left\{ \int_0^{2\pi} \left[\int f(v', t') f(w', t') \, d\mathbf{W} \right] d\epsilon \right\} d\kappa$$

$$B_{\kappa_0} = \frac{1}{4\pi} \int_0^{\kappa_0} \phi(\kappa)(\sin \kappa) \tag{26}$$

$$\times \left\{ \int_0^{2\pi} \left[\int f(v', t') f(w', t') - f(v, t') f(w, t') \, d\mathbf{W} \right] d\epsilon \right\} d\kappa$$

A_{κ_0} is not necessarily bounded when $\kappa_0 \to 0$ and if further $f(v', t')f(w', t') > 0$, then A_{κ_0} tends to $+\infty$ when $\kappa_0 \to 0$. On the contrary, due to the assumption $\phi_1 < \infty$, we see that $B_{\kappa_0} \to 0$, although the sign of B_{κ_0} is in general not known. In the following we always assume B_{κ_0} as being negligible compared to A_{κ_0} and we neglect B_{κ_0} in the discussion.

First we show that for finite v values, there exists a finite, v-dependent interval such that $f(v, t) > 0$. For $t = 0$ in Eq. (25), the bracket on the rhs is zero; since this bracket is a continuous t function, there always exists a finite $[0, t]$ interval where its modulus is less than $f(v, 0)$. This argument does not work for an infinite v value, because $f(v, 0)$ itself goes to zero. If we expand Eq. (25) around $t = 0$, we obtain

$$[\exp(\phi_{\kappa_0} t)] f(v, t) \simeq f(v, 0) + t[A_{\kappa_0}(v, 0) + B_{\kappa_0}(v, 0)] + O(t^2)$$

and we see that the two first-order terms are positive if $f(v, 0) > 0$ and B_{κ_0} negligible. Second, we show that the positivity property of f propagates forward in time. Let us assume $f(v, t) > 0$ for $t < t_0$; then in Eq. (25) the bracket is positive for $t = t_0$ and there exists an interval $t_0 + \Delta t_0$ (v dependent) such that the modulus of the bracket is less than $f(v, 0)$ using continuity in t; the sum of the two terms on the rhs of Eq. (25) leads to $f(v, t_0 + \Delta t_0) > 0$.

Similarly, one can show that the nonpositivity property of f propagates backward in time and must have appeared at a later time at v infinite. We assume $f(v, t) > 0$ for $t = 0$ and for a small Δt interval. On the other hand, we assume $f(v_0, t_0) < 0$. Using continuity in t of the second term on the rhs of Eq. (25), we see that there exists necessarily $t_1 < t_0$ and v_1 such that $f(v_1, t_1) < 0$. Continuing and always comparing the finite first term $f(v, 0)$ of the rhs of Eq. (25) with the second term, we define a sequence $t_2 < t_1 < t_0$ and $v_2, f(v_2, t_2) < 0$, $t_3 < t_2$, $f(v_3, t_3) < 0, \ldots$, and we see that the only possibility is that a negative part of $f(v, t)$ has appeared for infinite v value at time t_{lim} less than t_0.

The only escape to positivity for $t > t_0$ is the appearance of a negative part at infinite v value. Let us assume that $f(v, 0)$ does not oscillate for high velocities; for example, $f(v, 0)$ decreases monotonically to zero for large v.

Remembering that B_{κ_0} is small for small κ_0, we get from Eq. (25)

$$[\exp(\phi_{\kappa_0}t)][f(v_1,t)-f(v_2,t)]$$
$$\simeq [\exp(\phi_{\kappa_0}t_0)][f(v_1,t_0)-f(v_2,t_0)]$$
$$+\int_{t_0}^{t} dt'\,[\exp(\phi_{\kappa_0}t')][A_{\kappa_0}(v_1,t')-A_{\kappa_0}(v_2,t')]$$

If, at $t_0=0, f(v_1,0)-f(v_2,0)>0$ for $v_1<v_2$, then $f(v_1,t)-f(v_2,t)>0$ for a small time interval around 0 which depends on v_1 and v_2. The same holds for a small interval around t_0 if $f(v_1,t_0)-f(v_2,t_0)>0$. So, monotonicity at $t=t_0$ implies monotonicity at further times, but the answer is not complete, as we cannot prove that $f(v_2,t)$ remains positive. However, the above argument, although incomplete, contradicts the appearance of a negative tail at infinite v values, which would require for sufficiently large v_1 and v_2, $v_1<v_2$, that $f(v_1,t)-f(v_2,t)<0$.

5. GENERALIZATION TO d-DIMENSIONAL SYSTEMS

The above results may be extended with minor modifications to d-dimensional fluids. We just indicate in this section the main differences.

The Boltzmann equation (1) now reads

$$\frac{\partial f}{\partial t}(v,t)=\frac{1}{S_d}\iint [f(v',t)f(w',t)-f(v,t)f(w,t)]\phi^{(d)}(\kappa)\,d^d\mathbf{W}\,d\Omega_d \tag{27}$$

where S_d is the surface of the d-dimensional unit sphere, $d^d\mathbf{W}$ stands for the integration over the d-dimensional vector $\mathbf{W}$, and

$$d\Omega_d=(\sin\kappa)^{d-2}(\sin\epsilon)^{d-3}(\sin\epsilon_1)^{d-4}\cdots(\sin\epsilon_{d-4})\,d\kappa\,d\epsilon\,d\epsilon_1\cdots d\epsilon_{d-3}$$

is the d-dimensional solid angle expressed as a function of the $d-1$ polar angles κ, ϵ, $\epsilon_1,\ldots,\epsilon_{d-3}$, $0<\kappa$, ϵ, $\epsilon_1,\ldots,\epsilon_{d-4}<\pi$, $0<\epsilon_{d-3}<2\pi$, of the diffusion direction $\mathbf{V}'-\mathbf{W}'$ with respect to the incidence direction $\mathbf{V}-\mathbf{W}$. Expressions (1) for $(v')^2$ and $(w')^2$ still hold, but the normalized moments $M_n(t)$ are defined as

$$M_n(t)=2^{-n}\Gamma(d/2)[\Gamma(n+d/2)]^{-1}\int f(v,t)v^{2n}\,d^d\mathbf{V} \tag{28}$$

although the Sonine moments $b_n(t)$ are still given by (4).

The proof is slightly different from that for the three-dimensional case, and is indicated in Appendix D for $d>3$. For $d=2$, simplifications occur because there is only one polar angle, but the approach is the same and we shall not indicate it. We get for the M_n and b_n the nonlinear systems

$$\frac{d}{dt} M_n = \sum_{k=0}^{n} M_k M_{n-k} \tilde{B}_{k,n} C_n{}^k \tag{29a}$$

$$\frac{d}{dt} b_n = \sum_{k=0}^{n} b_k b_{n-k} \tilde{B}_{k,n} C_n{}^k \tag{29b}$$

where

$$\begin{aligned}
\tilde{B}_{k,n} &= \frac{2\pi^{-1/2}\Gamma(d/2)}{\Gamma((d-1)/2)} \\
&\quad \times \int_0^\pi \frac{1}{2} \phi^{(d)}(\kappa) \sin^{d-2} \kappa \left(\cos\frac{\kappa}{2}\right)^{2n-2k} \left(\sin\frac{\kappa}{2}\right)^{2k} d\kappa \qquad \text{if} \quad k \neq 0 \\
&= \frac{2\pi^{-1/2}\Gamma(d/2)}{\Gamma((d-1)/2)} \\
&\quad \times \int_0^\pi \frac{1}{2} \phi^{(d)}(\kappa) \sin^{d-2} \kappa \left[\left(\cos\frac{\kappa}{2}\right)^{2n} - 1\right] d\kappa \qquad \text{if} \quad k = 0
\end{aligned} \tag{29c}$$

plays the same role as $B_{k,n}$ if in all equations we make the substitution

$$\phi^{(3)}(\kappa) = \frac{2\pi^{-1/2}\Gamma(d/2)}{\Gamma((d-1)/2)} \phi^{(d)}(\kappa) \sin^{d-3} \kappa \tag{30}$$

the existence condition $\phi_1 < \infty$ being replaced by

$$\int_0^{\text{const}} \phi^{(d)}(\kappa) \sin^d \kappa \, d\kappa < \infty$$

Formulas (28)–(30) are still valid for $d = 2$.

It is known[3–6] that the function

$$F(x = v^2/2, t) = (2\pi)^{d/2} (\exp \tfrac{1}{2} v^2) f(v, t) \tag{31}$$

has a Sonine expansion in powers of generalized Laguerre polynomials $L_n^{(\alpha)}(v^2/2)$, the coefficients being the b_n. This may be seen in the following way, already used in Ref. 5:

Let $G(\xi, t) = \sum_{n \geq 0} \xi^n M_n(t)$ be the generating function. From (4), G may be rewritten in terms of the b_n

$$G(\xi, t) = \sum_{n \geq 0} \frac{\xi^n}{(1-\xi)^{n+1}} b_n(t)$$

and from (28) and (31), we have also

$$G(\xi, t) = 2\pi^{d/2} \int_0^\infty v^{d-1} f(v, t)\, dv \sum_{k \geq 0} \frac{(\frac{1}{2}v^2 \xi)^k}{\Gamma(k + d/2)}$$

$$= \int_0^\infty dx\, F(x, t) e^{-x} x^{(d-2)/2} \sum_{k \geq 0} \frac{(x\xi)^k}{\Gamma(k + d/2)}$$

Expanding $F(x, t)$ as $\sum_{n \geq 0} (-)^n b_n(t) l_n(x)$, where the $l_n(x)$ are yet unknown, we get the condition

$$\int_0^\infty dx\, x^{(d-2)/2} e^{-x} l_n(x) \sum_{k \geq 0} \frac{(x\xi)^k}{\Gamma(k + d/2)} = (-)^n \frac{\xi^n}{(1 - \xi)^{n+1}}$$

$$= (-)^n \sum_{l \geq 0} \xi^{n+l} \frac{(n + l)!}{n!\, l!}$$

or

$$\int_0^\infty dx\, e^{-x} \frac{l_n(x)}{\Gamma(k + d/2)} x^{(d-2)/2 + k} = 0 \qquad \text{if } \ k < n$$

$$= (-)^n C_k^{\,n} \qquad \text{if } \ k \geq n$$

whence[11]

$$l_n(x) = L_n^{(\alpha)}(x)$$

where α is a function of dimensionality

$$\alpha = (d - 2)/2 \tag{32}$$

Introducing the constraints of normalization and energy conservation ($b_0 \equiv 1, b_1 \equiv 0$) and setting $b_n = a_{n-2}$, we get the alternate expression for $F(x, t)$ corresponding to Eqs. (6)

$$F(x, t) = 1 + \sum_{n=0}^{\infty} (-)^n a_n(t) L_{n+2}^{(\alpha)}(x) \tag{33a}$$

$$\frac{d}{dt} a_n + \tilde{\beta}_n a_n = \sum_{m=0}^{n-2} a_m a_{n-2-m} \tilde{B}_{m+2,n+2} C_{n+2}^{m+2} \tag{33b}$$

with $\tilde{\beta}_n = -\tilde{B}_{0,n+2} - \tilde{B}_{n+2,n+2}$, and the system (33) is *identical* to (6) [$\beta_n = \tilde{\beta}_n$, $B_{m+2,n+2} = \tilde{B}_{m+2,n+2}$] when (30) holds.

Now the considerations of Section 3 are unchanged, provided the moments ϕ_k of the function $\phi^{(3)}$ are now replaced by

$$\tilde{\phi}_k = \int_0^\pi \phi^{(d)}(\kappa) \sin^{d-2+2k}\kappa\, d\kappa \Big/ \int_0^\pi \sin^{d-2+2k}\kappa\, d\kappa$$

Expressions for the generating functions for the Laguerre polynomials were already written in their general form in Eqs. (11)–(12).

Similarly, the proof of the existence of an infinite number of positive solutions of the BE holds (Section 4). Actually, the dependence on dimensionality is already included in Appendix C, where the main bounds are derived. The only modification appears in the behavior of $\phi(\kappa)$ at small angles, and Eq. (18) is replaced in the general case by

$$(\sin \kappa)^{d-3}\phi^{(d)}(\kappa) < C(\sin \tfrac{1}{2}\kappa)^{-2\eta}$$

the exponent η playing the same role as above. Again, there are infinitely many positive solutions of the BE which are not generalizations of that of Bobylev.

6. NUMERICAL CALCULATIONS

In this section, we present some numerical results using a function $\phi(\kappa)$ singular at the origin. We were especially interested in the study of a long-tail effect found by Krook and Wu[2]; however, they used a truncated BE and their effect could have been the result of an unjustified approximation. However, recently, a similar behavior was observed by Tjon for a $2d$ fluid and an energy-dependent collision frequency.[8] Roughly, depending on initial conditions, the distribution function $f(v, t)$ may relax to the Maxwell equilibrium function either in a monotonic way or with a tail population which is much more important than for $f(v, \infty)$: i.e., high-energy particles may be more numerous than in the equilibrium case, leading, for example, to higher reaction rates than expected. Using various initial conditions, we observed the transition between these two relaxation modes. Dimensionality and $\phi(\kappa)$ do not seem to play a great role, but we were not able to characterize the precise criteria for getting one or the other behavior; however, we have noticed some features favorable for one mode or for the other.

For our numerical experiments, we have chosen the Maxwellian case (potential $\simeq r^{-4}$ if $d = 3$); the diffusion angle is expressed as an elliptic integral[6]

$$\kappa(u) = \pi - (1-u)^{1/2} \int_0^\pi \frac{d\theta}{(1 + u\cos^2\theta)^{1/2}}$$

$$\phi(\kappa) \sin \kappa \frac{d\kappa}{du} = (1+u)u^{-3/2}$$

where u depends on the impact parameter. At small angles $\kappa \simeq u$ and $\phi(\kappa) \simeq u^{-5/2}$. More precisely, the normalized moments ϕ_k are

$$\phi_k = \frac{(2k+1)!}{2^{2k}k!\,k!} \int_0^1 du\,(1+u)u^{-3/2} \sin^{2k} \kappa(u) \tag{34}$$

Figures 1 and 2 show the evolution in time of the reduced function $F(x, t)$ for several initial conditions $F(x, 0)$. We use expansion (6a) up to the 12th term a_{12} and integrate the truncated system (6b). In Figs. 1a and 1b, $F(x, 0)$ defines a fundamental positive solution, although in situation 2, $a_n(0) \neq 0 \forall n$; this latter example was derived from the generating functions of Eq. (12) and we have even allowed $F(x, 0)$ to be negative for small x values but not for large enough x values; $F(x, 0)$ then has no physical meaning, though $F(x, t)$ is a solution of the BE; nevertheless, positivity for all x is rapidly

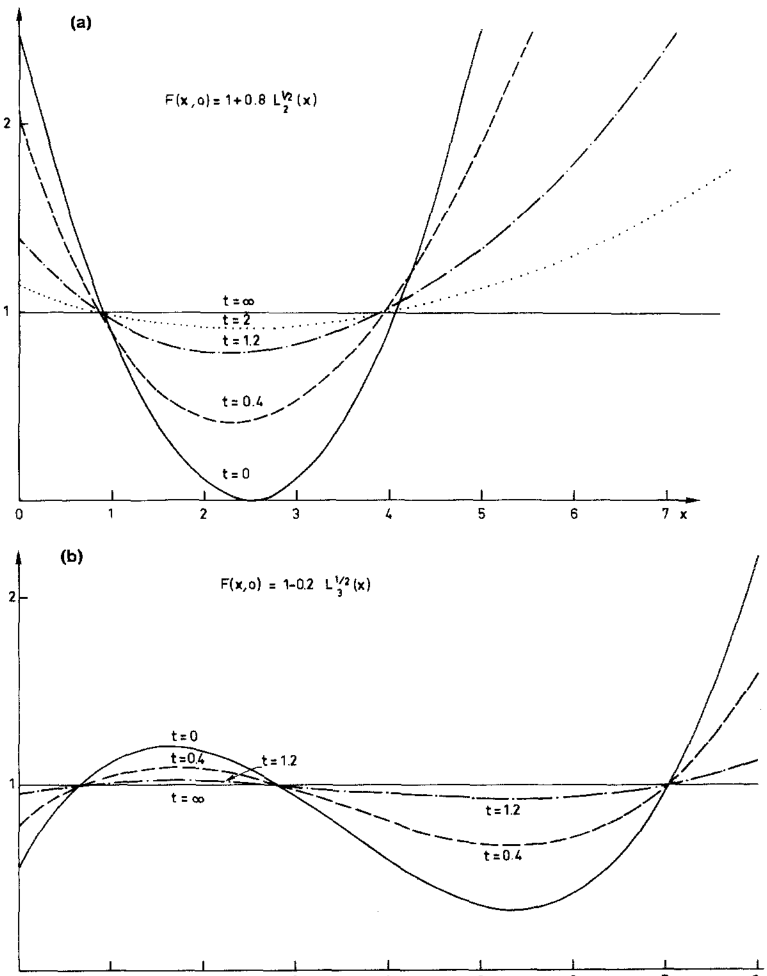


Fig. 1. Evolution in time for a fundamental positive solution and a 3d fluid. (a) $F(x, 0) = 1 + 0.8L_2^{1/2}(x)$. (b) $F(x, 0) = 1 - 0.2L_3^{1/2}(x)$.

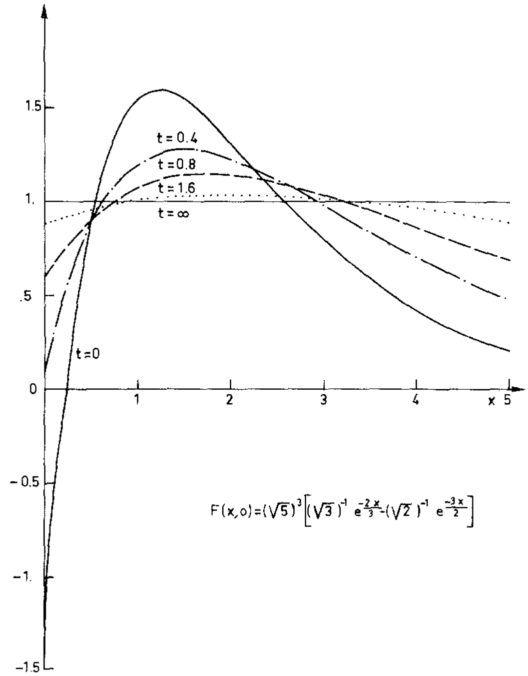


Fig. 2. Evolution in time of a solution which violates positivity at $t = 0$ for small x values; positivity is already restored for $t = 0.4$.

restored and in all cases $F(x, t) \to 1$ for long times, ensuring the Maxwellian behavior for $f(v, t)$. These initial conditions were already used in the isotropic case,[5] but it is not important, as they evolve differently. The convergence here is more rapid, ϕ_1, which gives the dominant behavior, is of the order of 3 ($\phi_1 \equiv 1$ in the isotropic case). There is no problem in extending this kind of calculation to fluids with higher dimensionality. We did it for $d = 4$, 6, and initial conditions given by (11), but the results are similar to the previous ones and we do not present them here.

The examples above correspond to a monotonic convergence to equilibrium. Tjon's effect does not exist in the fundamental positive solution (cf. ex. 1), since the high-energy behavior is dominated by the Laguerre polynomial [$F(x, 0) > 1$ for large x]; on the other hand, we could expect it in examples following from Eq. (12) (cf. ex. 2), but we did not see it. It seems that a physically reasonable distribution can be characterized in the following way. Let us call x_m the largest zero of $F(x, 0) - 1$:

(i) For $x > x_m$, $F(x, 0)$ is always less than one.

(ii) Consequently, due to the conservation of M_0, for $x < x_m$, $F(x, 0)$ will in general be greater than one, but not necessarily for all x, because $F(x, 0) - 1$ can have other zeros.

We look at the way $F(x, t) \to_{t\to\infty} 1$ for large $x > x_m$, either always from below, i.e., $F(x, t) \leqslant 1\ \forall t \in [0, \infty]$ or from above: $F(x, t) \geqslant 1$ for t greater than some critical value t_x. In the second mode, when t increases, $F(x, t)$ for x fixed higher than x_m must first cross the value one for $t = t_x$ and for $t \geqslant t_x$ reaches a maximum greater than one (enhancement) and relaxes to one when $t \to \infty$.

In order to study the transition from one mode to another, we choose families of $F(x, 0)$ depending smoothly on some parameters that we vary continuously. Although the transition, if it exists, is obtained continuously, there exists for the parameter an intermediate interval where the second mode is not very visible. Moreover, the effect is interesting only if the enhancement is appreciable. We especially focused on families of initial conditions generated from identity (11) with $d_1 = -1$, z and the other d_k being arbitrary. Tjon's

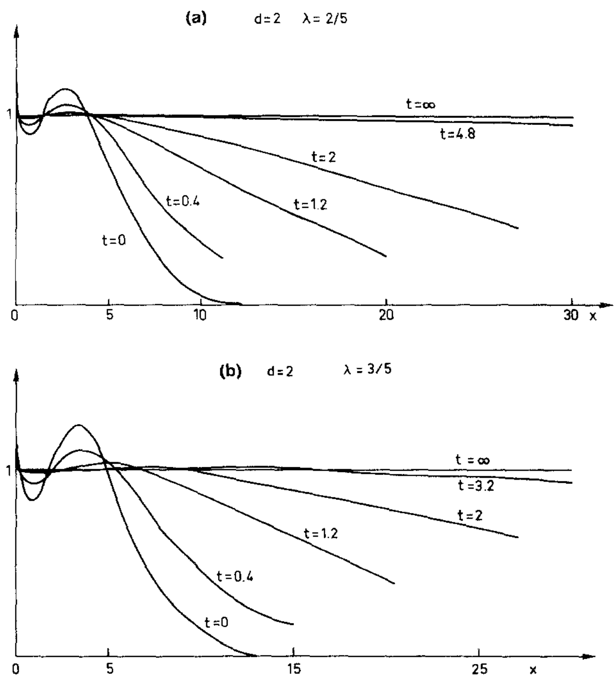


Fig. 3. Study of the transition in the one-parameter family (36) for a 2d gas. (a) $\lambda = 0.4$, the behavior is similar to that of the previous examples. (b) $\lambda = 0.6$ (transition region), the last zero moves to the right. (c) $\lambda = 0.9$, nonuniform relaxation mode. (d) Plot of $F(x, t)$ as a function of t for several values of x at $\lambda = 0.4$ (case a) and $\lambda = 1$ (case c).

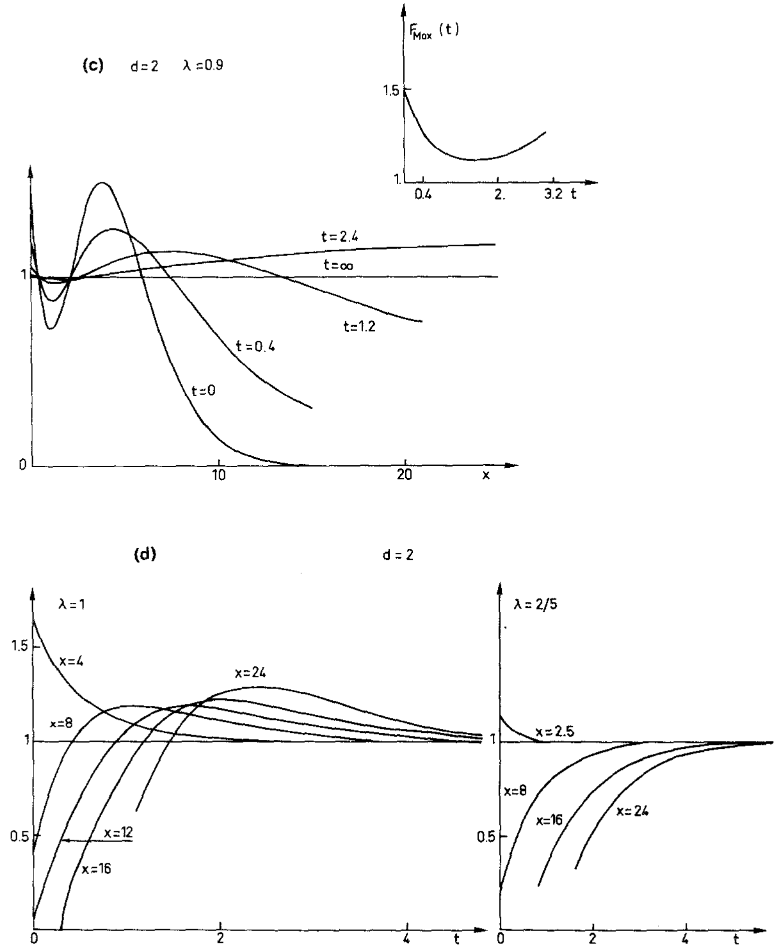


Fig. 3. Continued.

effect may be seen when *the exponential decrease is not too large and is still slowed by a polynomial with large coefficients. The existence of two extrema* seems to be important, since this mode of relaxation does not exist in families (12) nor in the BKW family. For the latter case, this can be proved analytically in a simple way: the BKW similarity solution reads

$$F(x, t) = \frac{1}{(1-z)^{\alpha+1}}\left[\exp\left(-\frac{z}{1-z}x\right)\right]\left[1 - \frac{z}{1-z}(\alpha+1) + \frac{zx}{(1-z)^2}\right] \quad (35)$$

where $z = z(t) = z(0)\exp(-\phi_1 t/6)$ and $F(x, 0)$ is a peculiar case of family (11) with $d_k = 0\ \forall k \geqslant 2$ and $0 < z(0) \leqslant 1/(2+\alpha)$. It is enough to prove that $F(x, t) \leqslant 1\ \forall t$ if $x > 5 + 2\alpha$; setting $x = 3 + \alpha + \bar{x}$, we rewrite $F(x, t)$ as

$$F(x,t) = \left\{\frac{\exp\{-[(3+\alpha)/(1-z)]z\}}{(1-z)^{3+\alpha}}\right\}$$
$$\times \left\{\left[\exp\left(-\frac{z}{1-z}\bar{x}\right)\right][1 + z^2(2+\alpha) + z\bar{x}]\right\}$$

The first bracket is always less than 1, and so is the second bracket for $\bar{x} > 2 + \alpha$, whence the result. On the contrary, the effect exists when we introduce a second coefficient d_2, but no compact form is possible and the results are numerical.

A typical example is the family (for a 2d gas)

$$F_\lambda(x, 0) = e^{-x}[1 + \tfrac{1}{2}\lambda + 2x^2(1-\lambda) + \tfrac{1}{3}\lambda x^4], \qquad 0 < \lambda < 2.27 \tag{36}$$

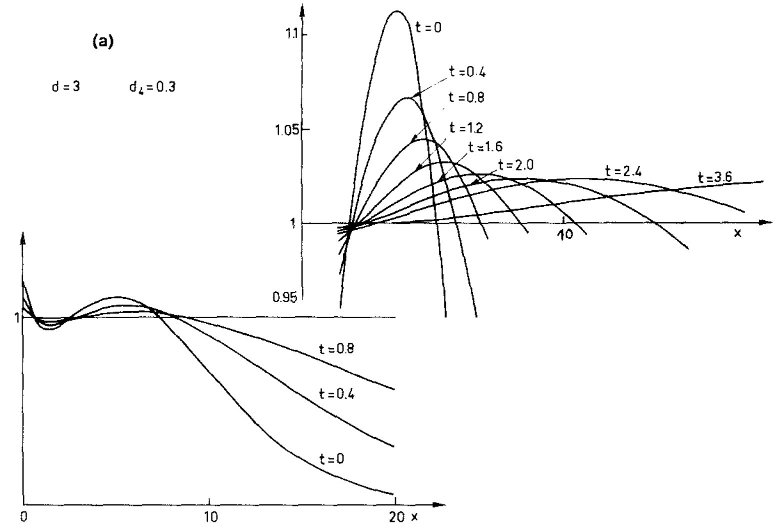
where $z = 1/2$, $d_2 = \frac{1}{2} + \lambda$, $d_3 = -\lambda$, $d_4 = \lambda/4$; with λ fixed, $F_\lambda(x, t)$ has two extrema and three intersections (zeros) with $F(x, \infty) \equiv 1$ (Fig. 3). When $\lambda < 0.5$, the abscissa of the extrema and zeros are stable as time increases and $F(x, t)$ goes uniformly to 1 (Fig. 3a). At $\lambda \simeq 0.5$, the largest zero begins to move to the right when $t \to \infty$, the other characteristic features being unchanged (Fig. 3b). At $\lambda = 0.7$ and greater, both the largest zero and the maximum go to infinity and the width of the bump becomes larger and larger (Fig. 3c) to preserve the correct normalization $\int_0^\infty e^{-x}F(x, t)\,dx = 1$. The convergence to 1 is no longer uniform, as can be seen by plotting $F(x, t)$ for a given x as a function of t (Fig. 3d). In the subcritical region $\lambda < 0.5$, $F(x, t)$ goes monotonically to 1 when time increases, from below or above. After the transition, two cases occur: either $x < x_m$, in which case $F(x, t)$ goes monotonically to 1 when $t \to \infty$; or $x > x_m$, and $F(x, t)$ passes through a maximum at intermediate t before decreasing to 1. Practically, there exists a transition zone ($0.5 < \lambda < 0.7$ here), since the deformation seems to be continuous.

We found the same effect in the family

$$F_\mu(x, 0) = \left(\frac{3}{2}\right)^{3/2} e^{-x/2}\left[\frac{5}{8} + \frac{9\times 35}{32}\mu + \frac{9}{16}x^2\left(\frac{2}{5} - 21\mu\right) + \frac{27}{32}\mu x^4\right] \tag{37}$$

with $z = 1/3$, $\mu = d_4/64$, and for a three-dimensional fluid, the transition zone being $0.20 < d_4 < 0.25$ (Fig. 4). Another example of the same behavior for $d = 4$ (or $\alpha = 1$) is given by the family (Fig. 5)

$$F_\nu(x, 0) = (\tfrac{5}{3})^2 e^{-2x/3}[\tfrac{1}{3} - 9\nu + (\tfrac{5}{3}x)^2(\tfrac{1}{9} + \tfrac{5}{2}\nu) - \tfrac{1}{6}\nu(\tfrac{5}{3}x)^5] \tag{38}$$



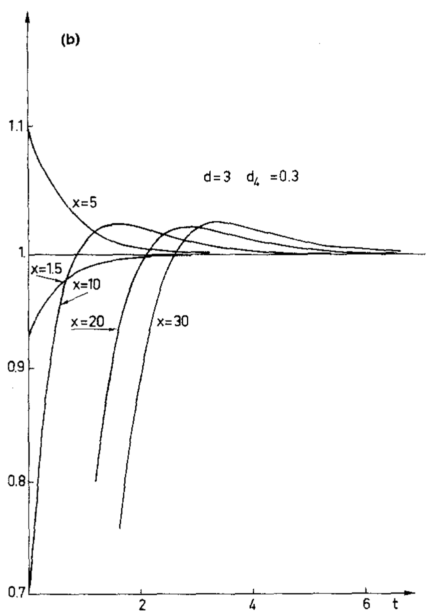


Fig. 4. Study of the effect in family (37) for a 3d gas. (a) Plot of $F_\mu(x, t)$ for several times above the critical value ($d_4 = 0.30$ or $\mu = 0.0047$). (b) Plot of $F_\mu(x, t)$ for $d_4 = 0.30$ as a function of t for several values of x.

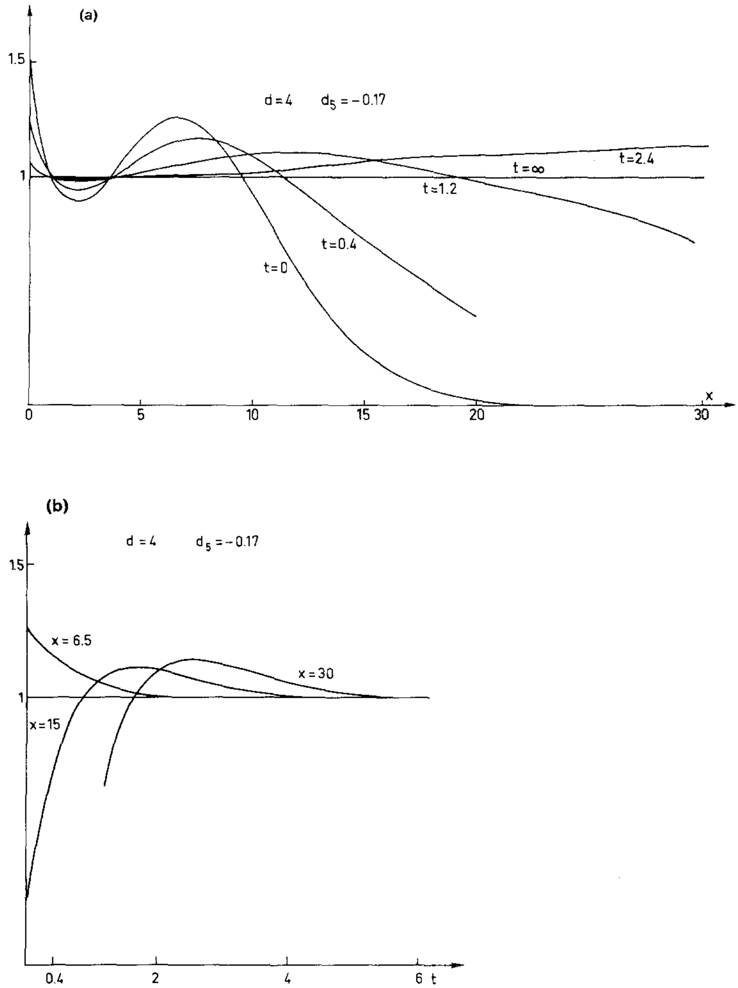


Fig. 5. The same as Figs. 4a and 4b, but for family (38) and a 4d gas; here $d_5 = -0.17$ (or $v = -0.0224$).

with $z = 2/5$, $v = (\frac{2}{3})^5 d_5$, $v < 0$, the transition zone being approximately $-0.15 < d_5 < -0.12$.

The transition may be systematically, but numerically, studied in the simplest case $d_1 = -1, d_k = 0, k \geqslant 3, z$ and d_2 arbitrary. Figure 6 corresponds to $d = 2$, or $\alpha = 0$; the positivity condition delimits a region in the (z, d_2) plane and the transition zone is a band, roughly $1.2 < d_2 < 1.5$, which does not seem to depend much on z. No effect exists for $z > 0.55$.

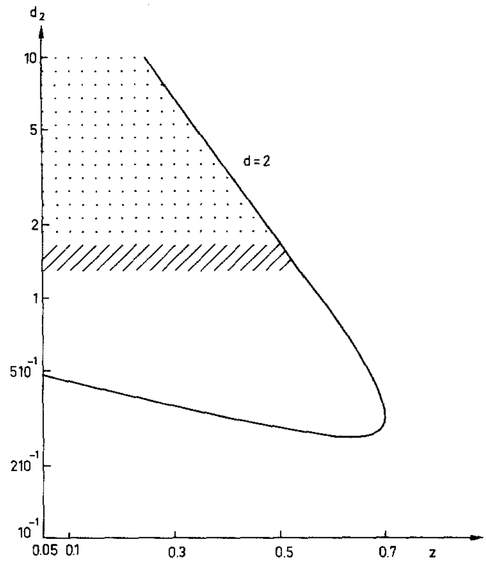


Fig. 6. Study of the critical zone in the (z, d_2) plane. The curve is the limit for positive initial conditions. The transition region is the dashed band, approximately $1.2 < d_2 < 1.5$; the effect is quite apparent in the dotted region.

APPENDIX A

A1. Proof that $\lambda(q, q', n) \equiv 0$ *unless* $q + q' = n$. We have

$$\lambda(q, q', n) = \frac{n!}{q!\,(q')!} \sum_{p=q+q'}^{n} \frac{(-1)^p}{(n-p)!} \frac{\bar{\lambda}(p, q, q')}{(p-q-q')!}$$

$$\bar{\lambda}(p, q, q') = \sum_{k=0}^{p-q-q'} B_{k+q,p} C_{p-q-q'}^{k}$$

We first consider $q \neq 0$; with the definition of $B_{k,n}$, Eq. (3), we obtain

$$\bar{\lambda} = \frac{1}{2} \int \phi(\kappa) \sin \kappa \left(\cos \frac{\kappa}{2} \right)^{2q} \left(\sin \frac{\kappa}{2} \right)^{2q'}$$

$$\times \left[\sum_{k=0}^{p-q-q'} \left(\cos \frac{\kappa}{2} \right)^{2(p-q-q'-k)} \left(\sin \frac{\kappa}{2} \right)^{2k} C_{p-q-q'}^{k} \right] d\kappa$$

Note that the sum in the bracket is 1 and substitute into λ:

$$\lambda = \frac{n!}{q!\,(q')!}(-1)^{q+q'}\frac{1}{2}\int \frac{\phi(\kappa)\sin\kappa(\cos\frac{1}{2}\kappa)^{2q}(\sin\frac{1}{2}\kappa)^{2q'}}{(n-q-q')!} \times \sum_{p=0}^{n-q-q'}(-1)^p C^p_{n-q-q'}\,d\kappa \equiv 0 \qquad \text{if} \quad n \neq q+q'$$

Second we consider $q = 0$,

$$\bar{\lambda} = \frac{1}{2}\int \phi(\kappa)\sin\kappa\left[-1 + \left(\cos\frac{\kappa}{2}\right)^{2q'}\sum_{k=0}^{p-q'}\left(\cos\frac{\kappa}{2}\right)^{2(p-q'-k)}\left(\sin\frac{\kappa}{2}\right)^{2k} C^k_{p-q'}\right]d\kappa$$

Note that the sum over k gives 1 and

$$\lambda = \frac{n!}{q!}\frac{(-1)^q}{2}\int \phi(\kappa)\left(\cos\frac{\kappa^{2q'}}{2} - 1\right)\sum_{p=0}^{n-q'}(-1)^p C^p_{n-q'}\,d\kappa \equiv 0 \qquad \text{if} \quad n \neq q'$$

A2. Derivation of Eq. (5b) using only sonine moments. Replacing $f(v, t)$ by its expansion (5a), we get

$$\sum_{n\geqslant 0}(-1)^n\frac{d}{dt}b_n(t)L_n^{(1/2)}\left(\frac{v^2}{2}\right)$$
$$= \frac{1}{4\pi}\sum_{m\geqslant 0}\sum_{m'\geqslant 0}(-1)^{m+m'}b_m(t)b_{m'}(t)$$
$$\times \int d\mathbf{W}\int \phi(\kappa)\sin\kappa\,d\kappa\,d\epsilon\exp\left(-\frac{w^2}{2}\right)(2\pi)^{-3/2}$$
$$\times\left[L_m^{(1/2)}\left(\frac{(v')^2}{2}\right)L_{m'}^{(1/2)}\left(\frac{(w')^2}{2}\right) - L_m^{(1/2)}\left(\frac{v^2}{2}\right)L_{m'}^{(1/2)}\left(\frac{w^2}{2}\right)\right]$$

Multiplying both sides by $L_n^{(1/2)}(v^2/2)\exp(-v^2/2)$ and integrating over $\mathbf{V}$, we get

$$(-1)^n\frac{db_n}{dt}\lambda_n = \frac{1}{4\pi}\sum_{m\geqslant 0}\sum_{m'\geqslant 0}(-1)^{m+m'}b_m(t)b_{m'}(t)$$
$$\times\int\phi(\kappa)\sin\kappa\,d\kappa\,d\epsilon\iint\frac{d\mathbf{V}\,d\mathbf{W}}{(2\pi)^{3/2}}$$
$$\times\exp\left[-\frac{(v^2+w^2)}{2}\right]L_m^{(1/2)}\left(\frac{v^2}{2}\right)L_{m'}^{(1/2)}\left(\frac{w^2}{2}\right)$$
$$\times\left[L_n^{(1/2)}\left(\frac{(v')^2}{2}\right) - L_n^{(1/2)}\left(\frac{v^2}{2}\right)\right]$$

where λ_n is the normalization factor for the $L_n^{(1/2)}$. We perform the integration over ϵ and θ (angle between **V** and **W**), using the identity

$$(4\pi)^{-1}\int_0^{\pi}\sin\theta\,d\theta\int_0^{2\pi}d\epsilon\,L_n^{(1/2)}\left(\frac{(v')^2}{2}\right)$$
$$=(vw\sin\kappa)^{-1}\left[L_{n+1}^{(-1/2)}\left(\frac{1}{2}\left(v\cos\frac{\kappa}{2}-w\sin\frac{\kappa}{2}\right)^2\right)\right.$$
$$\left.-L_{n+1}^{(-1/2)}\left(\frac{1}{2}\left(v\cos\frac{\kappa}{2}+w\sin\frac{\kappa}{2}\right)^2\right)\right]$$

From the definition

$$n!\,L_n^{(\alpha)}(x)=e^x x^{-\alpha}\frac{d^n}{dx^n}\left(e^{-x}x^{n+\alpha}\right)$$

and orthogonalization relations for the Laguerre polynomials, we verify that the summations on the rhs reduce to the m, m' such that $m+m'=n$; Eq. (5b) is then straightforward.

APPENDIX B

B1. We determine the expansions of the $B_{m,n}$ in terms of the ϕ_p. From the definitions (3)–(8) we obtain if $m<[n/2]-1$

$$B_{m+2,n+2}+B_{n-m,n+2}=\frac{1}{2}\int_0^{\pi}d\kappa\,\phi(\kappa)\frac{\sin\kappa^{2m+5}}{2^{2m+4}}P_{n-2m-2}(\sin\kappa)\,d\kappa \tag{B1}$$

$$P_q(\sin\kappa)=\left(\cos\frac{\kappa}{2}\right)^{2q}+\left(\sin\frac{\kappa}{2}\right)^{2q}$$
$$=\sum_0^{[q/2]}(-1)^p\frac{\sin\kappa^{2p}}{2^{2p}}\frac{q!}{p!}\frac{(q-p-1)!}{(q-2p)!} \tag{B2}$$

with $[q/2]=q/2$ if q is even and $[q/2]=(q-1)/2$ if q is odd. Substituting the expansion of P_{n-2m-2} into the rhs of (B1) and taking into account the definition (7) of the ϕ_p, we obtain the result (8). Since β_n is equal to $\frac{1}{2}\int\phi(\kappa)(\sin\kappa)(1-P_n)\,d\kappa$, we must subtract the constant term equal to 1 in the expansion (B2) of P_n. Consequently β_n can be obtained from the expression (8) for $B_{m+2,n+2}+B_{n-m,n+2}$ where we subtract the first term for $p=0$. Finally, if n is even,

$$B_{n/2+1,n+2}=2^{-n-2}\left[\frac{1}{2}\int\phi(\kappa)\sin\kappa^{n+2}\,d\kappa\right]$$

and we apply the definition of the ϕ_p, Eq. (7).

B2. We want to prove $\beta_n + \beta_{n-m-2} - \beta_n > 0\ \forall m \in [0, n-2]$ or equivalently from the definition of β_p and putting $u = (1 - \cos\kappa)/2$, $\phi(\kappa) = \bar{\phi}(u)$,

$$\int_0^1 du\, \bar{\phi}(u)[1 + u^{n+2} + (1-u)^{n+2} - u^{m+2} - (1-u)^{m+2} - u^{n-m} - (1-u)^{n-m}] > 0 \qquad \text{(B3)}$$

Let us denote the bracket by $z(u)$. It is sufficient to show that $z(u) > 0$ for $u \in [0, 1]$ and $m \in [0, n-2]$, $n \geqslant 2$. We obtain

$$\frac{dz}{du} = u(1-u)\left\{(m+2)\sum_{k=0}^{n-m-1}[(1-u)^{m+k} - u^{m+k}] + (n-m)\sum_{k=0}^{m+1}[(1-u)^{n-m+k-2} - u^{n-m-k-2}]\right\}$$

and $z'(\tfrac{1}{2}) = 0$, $z'(u) > 0$ for $u < \tfrac{1}{2}$; $z'(u) < 0$ for $u > \tfrac{1}{2}$. It follows that $z(u) \geqslant z(0) = 0$.

APPENDIX C

In this appendix we always consider the normalization λ_n associated with $L_{n+2}^{(\alpha)}$, $\alpha \geqslant 0$,

$$\lambda_n = \Gamma(n + 3 + \alpha)/\Gamma(n + 3) \qquad \text{(C1)}$$

C1. Bound on $[\lambda_n(\lambda_m\lambda_{n-m-2})^{-1}]^{1/2}$, $m \in [0, n-2]$, $n \geqslant 2$. We define $\delta_n{}^m = \lambda_m\lambda_{n-m-2}$ and remark that for n fixed, $p < (n-3)/2$, $\delta_n{}^p$ is increasing,

$$\delta_n{}^p - \delta_n^{p+1} = \frac{\alpha\Gamma(n-p+\alpha)\Gamma(p+3+\alpha)}{\Gamma(n-p+1)\Gamma(p+4)}(2p + 3 - n)$$

It follows that $\delta_n{}^m \geqslant \delta_n{}^0$ and finally

$$\left(\frac{\lambda_n}{\lambda_m\lambda_{n-m-2}}\right)^{1/2} \leqslant \Lambda_n = \left[\frac{2(\alpha+n+2)(\alpha+n+1)}{(n+1)(n+2)\Gamma(\alpha+3)}\right]^{1/2} \qquad \text{(C2)}$$

C2. Bound on $N(t) = \sum_{n \geqslant n_0} |a_n(t)|\lambda_n^{1/2}$ *if* $\phi(\kappa) < (\sin\tfrac{1}{2}\kappa)^{-2}$. We start with the representation (9b) of the solution $a_n(t)$ with the initial values $a_n(0)$, note from (19) that $B_{m+2,n+2}C_{n+2}^{m+2} < (m+2)^{-1}$, multiply both sides by $\lambda_n^{1/2}$, take the modulus of both sides, and bound the rhs:

$$n \leqslant 2n_0 + 1: \qquad |\sqrt{\lambda_n}a_n(t)| = |\sqrt{\lambda_n}a_n(0)| \exp(-\beta_n t)$$

$$n \geqslant 2n_0 + 2: \qquad |\sqrt{\lambda_n}a_n(t)| \leqslant [\exp(-\beta_n t)]\left[|a_n(0)\sqrt{\lambda_n}| + \int_0^t [\exp(\beta_n t')]\sqrt{\lambda_n} \times \sum_{m+p=n-2}(m+2)^{-1}|a_m(t')||a_p(t')|\, dt'\right] \qquad \text{(C3)}$$

From (C2) and the substitutions $\beta_n \to \beta_{n_0}$, $(\lambda_n/\lambda_m\lambda_p)^{1/2} \to \Lambda_n$, and

$$\Lambda_n/(m+2) \to \Lambda_{2n_0+2}/(m+2) \to \Lambda_{2n_0+2}/(n_0+2)$$

we obtain upper bounds on the rhs of (C3). Summing over n, we find a nonlinear inequality:

$$N(t)\exp(\beta_{n_0}t) \leqslant M(t) = N(0) + \frac{\Lambda_{2n_0+2}}{n_0+2}\int_0^t N_0{}^2(t')\exp(\beta_{n_0}t')\,dt'$$

which we integrate as in Ref. 5. We obtain: if

$$N(0) \leqslant \beta_{n_0}(n_0+2)(\Lambda_{2n_0+2})^{-1}$$

then

$$N(t) \leqslant \frac{N(0)\beta_{n_0}(n_0+2)}{\Lambda_{2n_0+2}} \times \{N(0) + [\beta_{n_0}(n_0+2)(\Lambda_{2n_0+2})^{-1} - N(0)]\exp(\beta_{n_0}t)\}^{-1} \qquad \text{(C4)}$$

Notice that if $\eta = 1$, $\beta_n \sim \log n$, although $B_{m+2,n+2}C_{n+2}^{m+2}$ remains bounded. If we do not consider λ_n and perform the same analysis, we obtain an upper bound for $\sum |a_n(t)|$ (where Λ_n disappears) if

$$\sum |a_n(0)| < (n_0+2)\beta_{n_0}$$

Returning to (C3), where

$$\sum (m+2)^{-1}|a_m|\,|a_p| < (n_0+2)^{-1}\,[\sum |a_n(t')|]^2$$

and using the bound for $\sum |a_n|$, we can integrate and obtain directly a bound on $|a_n(t)|$:

$$|a_n(t)| \leqslant [\exp(-\beta_n t)]|a_n(0)| + \frac{(n_0+2)^3\beta_{n_0}^4[\exp(-2\beta_{n_0}t) - \exp(-\beta_n t)]}{[\beta_{n_0}(n_0+2) - \sum |a_n(0)|]^2(\beta_n - 2\beta_{n_0})} \qquad \text{(C5)}$$

C3. Bound on $N(t)$ if $\eta < 2$. We start with the representation (9b) of $a_n(t)$, assume that $\phi(\kappa)$ satisfies the bound (18′), and introduce r, $0 < r < 1$, and R_{n_0} defined in (22). We obtain

$$n \leqslant 2n_0 + 1: \qquad |\sqrt{\lambda_n}a_n(t)| < |\sqrt{\lambda_n}a_n(0)|\exp(-r\beta_n t)$$

$$n \geqslant 2n_0 + 2: \qquad |\sqrt{\lambda_n}a_n(t)| < [\exp(-\beta_n t)]\Big\{|\sqrt{\lambda_n}a_n(0)| + \Lambda_{2n_0+2} \times \int_0^t \exp(\beta_n t') \sum_{\substack{m+p=n-2\\ m\geqslant n_0}} |a_m(t')\sqrt{\lambda_m}|\,|a_p(t')\sqrt{\lambda_p}| \times C_{n+2}^{m+2}B_{m+2,n+2}\,dt'\Big\} \qquad \text{(C6)}$$

We want to show that $|a_n(t)\sqrt{\lambda_n}| < [\exp(-r\beta_n t)]K_n$, where K_n is the solution of the recurrence equation

$$K_n = |a_n(0)|\sqrt{\lambda_n} + \frac{\Lambda_{2n_0+2}R_{n_0}}{1-r} \sum_{m+p=n-2} K_m K_p \tag{C7}$$

This is true for $n \leqslant 2n_0 + 1$. For $n \geqslant 2n_0 + 2$ let us show this relation by induction and assume it holds for $p = n_0, n_0 + 1, \ldots, n - 2$. We obtain from (C6), (C7), and (22)

$$\begin{aligned} &|\sqrt{\lambda_n}a_n(t)| - [\exp(-\beta_n t)]\sqrt{\lambda_n}|a_n(0)| \\ &\quad \leqslant \Lambda_{2n_0+2} \sum_m \int_0^t dt'\, K_m K_{n-m-2} C_{n+2}^{m+2} B_{m+2,n+2} \\ &\quad\quad \times \exp[(\beta_n - r\beta_m - r\beta_{m-n-2})t'] \end{aligned} \tag{C8}$$

Further,

$$\beta_n - r\beta_m - r\beta_{n-m-2} = (1-r)\beta_n + r(\beta_n - \beta_m - \beta_{n-m-2}) < (1-r)\beta_n$$

and if we integrate the rhs of (C8), taking into account (22), we see that it is bounded by

$$\Lambda_{2n_0+2}R_{n_0}(1-r)^{-1} \sum_m K_m K_{n-m-2}$$

and (C7) holds for n. It follows that

$$N(t) \leqslant \sum [\exp(-r\beta_n t)]K_n \leqslant [\exp(-r\beta_{n_0} t)] \sum K_n$$

Let us define $\mathscr{K} = \sum K_n$. From (C7) we get

$$\mathscr{K} = N(0) + \Lambda_{2n_0+2}R_{n_0}(1-r)^{-1}\mathscr{K}^2$$

and

$$\begin{aligned} N(t) &\leqslant [\exp(-r\beta_{n_0} t)]\{1 - [1 - 4\Lambda_{2n_0+2}N(0)R_{n_0}(1-r)^{-1}]^{1/2}\} \\ &\quad \times [2\Lambda_{2n_0+2}R_{n_0}(1-r)^{-1}]^{-1} \end{aligned} \tag{C9}$$

with the sufficient condition

$$\sum |a_n(0)|\sqrt{\lambda_n} = N(0) \leqslant (1-r)(4\Lambda_{2n_0+2}R_{n_0})^{-1} \tag{C10}$$

We note that the K_n can be explicitly determined. We define

$$\mathscr{K}(z) = \sum K_n z^n, \qquad N(t = 0, z) = \sum |a_n(0)\sqrt{\lambda_n}|z^n$$

and assume that (C10) holds. It follows that $N(t = 0, z)$ is an entire series with a finite radius of convergence. Then K_n can be determined from the expansion near $z = 0$:

$$\sum K_n z^n \equiv \left\{1 - \left[1 - 4\Lambda_{2n_0+2} R_{n_0}(1-r)^{-1} \sum |a_n(0)| z^n\right]^{1/2}\right\} \times [2\Lambda_{2n_0+2} R_{n_0}(1-r)^{-1}]^{-1} \tag{C11}$$

APPENDIX D.

Proof of the moment equations (29) for $d > 3$. Multiplying the BE (27) by $2^{-n}\Gamma(d/2)[\Gamma(n + d/2)]^{-1} v^{2n}$ and integrating over $\mathbf{V}$, we get

$$\frac{d}{dt} M_n = \frac{1}{S_d} 2^{-n}\Gamma\left(\frac{d}{2}\right)\left[\Gamma\left(n + \frac{d}{2}\right)\right]^{-1} \times \iint d^d\mathbf{V}\, d^d\mathbf{W}\, d\Omega_d\, \phi^{(d)}(\kappa) f(v,t) f(w,t)[(v')^{2n} - v^{2n}] \tag{D1}$$

where integrations on $\mathbf{V}$, $\mathbf{W}$, and $d\Omega_d$ are now d-dimensional and

$$(v')^2 = v^2 \frac{1 + \cos\kappa}{2} + w^2 \frac{1 - \cos\kappa}{2} + vw \sin\kappa \sin\theta \cos\epsilon$$

where θ is again the angle between $\mathbf{V}$ and $\mathbf{W}$.

The angular dependence of $\mathbf{V}$ can be removed, giving a multiplying constant S_d, and that for $\mathbf{W}$ is reduced to the polar angle θ; integrating over the $d - 2$ other polar angles, we get

$$\int d^d\mathbf{W} f(w,t)[(v')^{2n} - v^{2n}] = \int_0^\infty dw\, w^{d-1} f(w,t) \int_0^\pi \sin^{d-2}\theta[(v')^{2n} - v^{2n}] \frac{2\pi^{d-1/2}}{\Gamma[(d-1)/2]}$$

Similarly

$$\int d\Omega_d[(v')^{2n} - v^{2n}] = \int_0^\pi \phi^{(d)}(\kappa) \sin^{d-2}\kappa\, d\kappa \int_0^\pi \sin^{d-3}\epsilon[(v')^{2n} - v^{2n}] \frac{S_d}{\pi}\left(\frac{d}{2} - 1\right)$$

whence

$$\frac{d}{dt} M_n = \frac{d/2 - 1}{2^{n-2}} \frac{\pi^{d-3/2}}{\Gamma(n + d/2)\Gamma[(d-1)/2]} \int_0^\pi \phi^{(d)}(\kappa) \sin^{d-2} \kappa \, d\kappa$$

$$\times \int_0^\infty v^{d-1} f(v, t)\, dv \int_0^\infty w^{d-1} f(w, t)\, dw$$

$$\times \int_0^\pi \sin^{d-2} \theta \int_0^\pi \sin^{d-3} \epsilon [(v')^{2n} - v^{2n}] \tag{D2}$$

The last two integrations are easily performed. From

$$(v')^{2n} = \left(v^2 \frac{1 + \cos\kappa}{2} + w^2 \frac{1 - \cos\kappa}{2} + vw \sin\kappa \sin\theta \cos\epsilon \right)^n$$

$$= \sum_{k=0}^{n} C_n^k \left(v^2 \frac{1 + \cos\kappa}{2} + w^2 \frac{1 - \cos\kappa}{2} \right)^{n-k} (vw \sin\kappa \sin\theta \cos\epsilon)^k$$

and

$$\int_0^\pi \sin^{d-2+l} \theta \, d\theta \int_0^\pi \cos^l \epsilon \sin^{d-3} \epsilon$$

$$= 0 \qquad \text{if } l \text{ is odd}$$

$$= \frac{\Gamma(1/2)\Gamma(d/2 - 1)\Gamma[(l+1)/2]}{\Gamma[(l+d)/2]} \qquad \text{if } l \text{ is even}$$

we get, expanding

$$\left(v^2 \frac{1 + \cos\kappa}{2} + w^2 \frac{1 - \cos\kappa}{2} \right)^{n-k}$$

in powers of v,

$$\int_0^\pi d\theta \sin^{d-2} \theta \int_0^\pi d\epsilon \sin^{d-3} \epsilon (v')^{2n}$$

$$= \sum_{k=0}^{[n/2]} 2^{2k} C_n^{2k} \sum_{j=0}^{n-2k} C_{n-2k}^j v^{2j+2k} \left(\cos\frac{\kappa}{2} \right)^{2j+2k}$$

$$\times w^{2n-2k-2l} \left(\sin\frac{\kappa}{2} \right)^{2n-2k-2l} \frac{\Gamma(1/2)\Gamma(d/2-1)\Gamma(k+1/2)}{\Gamma(k+d/2)}$$

Setting $m = j + k$ and reordering, we have

$$\int_0^\pi d\theta \sin\theta^{d-2} \int_0^\pi d\epsilon \sin^{d-3}\epsilon[(v')^{2n} - v^{2n}]$$

$$= \pi\Gamma\left(\frac{d}{2} - 1\right) \sum_{m=0}^{n} C_n^m \left[\left(\cos\frac{\kappa}{2}\right)^{2m}\left(\sin\frac{\kappa}{2}\right)^{2n-2m} - \delta_{m0}\right].$$

$$\times\, v^{2m} w^{2n-2m} \sum_{k=0}^{\inf(m,n-m)} \frac{1}{k!} \frac{1}{\Gamma(k+d/2)} \frac{m!}{(m-k)!} \frac{(n-m)!}{(n-m-k)!}$$

This latter sum is actually the hypergeometric[11]

$$\frac{1}{\Gamma(d/2)}\, {}_2F_1\left(-m, m-n; \frac{d}{2}; 1\right) = \frac{\Gamma(n+d/2)}{\Gamma(m+d/2)\Gamma(n-m+d/2)}$$

Collecting all these results, we get

$$\frac{d}{dt} M_n = \frac{4\pi^{(d-1)/2}\Gamma(d/2)}{2^n\Gamma[(d-1)/2]} \sum_{m=0}^{n} C_n^m \int_0^\infty \frac{v^{2m+d-1}}{\Gamma(m+d/2)} f(v,t)\, dv$$

$$\times \int_0^\infty \frac{w^{2n-2m+d-1}}{\Gamma(n-m+d/2)} f(w,t)\, dw$$

$$\times \int_0^\infty \phi^{(d)}(\kappa) \sin^{d-2}\kappa\left(\cos^{2m}\frac{\kappa}{2}\sin^{2n-2m}\frac{\kappa}{2} - \delta_{m0}\right)$$

whence we obtain Eq. (29a) using again definition (28) of the M_n. Equation (29b) for the b_n is then derived as in the $d = 3$ case, using Eq. (4), which is independent of the dimensionality (Appendix A).

ACKNOWLEDGMENT

The authors are very grateful to Claudine Verneyre, who performed all the numerical calculations.

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